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Nonlinear extreme ground effect on thin wings of arbitrary aspect ratio

By E. O. TUCK

Department of Applied Mathematics, University of Adelaide,
GPO Box 498, Adelaide, South Australia 5001

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Air flows past a fixed thin body of a general planform, at a non-uniform small clearance from a plane ground surface. The flow beneath the body is described by a linear two-dimensional partial differential equation, in which the clearance appears as an input non-constant coefficient. Solutions are required subject to separate leading-edge and (nonlinear) trailing-edge boundary conditions, at the edge contour of the planform. The transition points between leading and trailing edge are not necessarily at the lateral extremities of this contour, and are to be determined as part of the solution. As an illustration, a solution is obtained for a circular planform with an exponentially varying clearance. The general problem is relevant to vehicle aerodynamics, especially for racing cars, and some qualitative discussion of the nature of the negative-lift ground-effect problem for such vehicles, and of the effect of 'skirts', is presented here.

1. Introduction

Ground effect enhances lift, be it positive or negative. Published studies (e.g. Strand, Royce & Fujita 1962; Widnall & Barrows 1970; Tuck 1980; Newman 1982) emphasize positive angle of attack, and hence positive (away from ground) net force. However, the influence of the ground plane in augmenting the toward-ground force on wings at negative angle of attack can be even greater, and recent applications to automobile aerodynamics, especially racing cars (Wise 1979) have illustrated this negative-lift phenomenon.

If one uses a *linearized* approach, as in Widnall & Barrows (1970), in which the flow about a body fixed in a uniform stream U is assumed *everywhere* a small perturbation of that stream, there is complete antisymmetry between positive and negative angle of attack. Although such linearized theories are of considerable value, and are further discussed here, we are more interested in nonlinear analysis, in which, at least in the small-gap region between the lower surface of the wing and the ground plane, the flow velocity is *not* close to that of the uniform stream.

The nature of the asymmetry between positive and negative angle of attack is illustrated easily by considering properties of such a nonlinear gap flow in the high-aspect-ratio case, as studied in Tuck (1981). Then the flow in the gap is determined by one-dimensional continuity (i.e. velocity inversely proportional to clearance), subject to smooth trailing-edge detachment, which demands freestream velocity and pressure at the trailing edge.

Then, at positive angle of attack, the velocity and pressure can vary between *stagnation* and *freestream* values under the wing, and the net positive lift is bounded above by the product of the velocity head $\frac{1}{2}\rho U^2$ and the area of the planform. This bound would be achieved only in the limit when the trailing edge touches the ground, and hence stagnation conditions apply everywhere in the gap. The maximum lift is notably independent of the angle of attack, and hence in practice potentially many times that achievable without ground effect, which is effectively proportional to the (small) angle of attack.

On the other hand, there is no lower bound on negative lift, according to this simple theory. That is, we can cause the pressure to take as great a negative value as we please in a one-dimensional channel, by letting the clearance become infinitesimally small at some station ahead of the trailing edge. The fluid has nowhere else to go, and must pass through this tight constriction at high speed, and hence at low pressure. In principle, then, we appear to be able to generate an arbitrarily large down force, a prospect that has been appreciated and to a large extent realized by Formula I racing car designers, in recent years, especially for vehicles fitted with so-called 'skirts'.

In practice, the down force is limited by real-fluid and finite-aspect-ratio considerations, the latter of which concerns us here; effects of viscosity are considered by Tuck & Bentwich (1983). If the wing is of finite span, and has no skirts, the flow in general is three- rather than two-dimensional, and the flow in the gap region is two- rather than one-dimensional, and few studies have been made of such flows. For the linearized problem, Widnall & Barrows (1970) made some computations for semi-elliptical planforms. More recently Newman (1982) solved the nonlinear problem in the *low*-aspect-ratio limit. The aim of the present paper is to formulate and discuss the nonlinear boundary-value problem for the flow in the gap at arbitrary aspect ratio, preparing the way for numerical solution of practical problems of this nature.

An interesting aspect of the work of Newman is that associated with the effective boundary condition at the edge contour of the wing planform. One portion of this contour, generally in the forward half, is a 'leading' edge, while the remainder is a 'trailing' edge, and different boundary conditions apply at these two edges. Transition between leading and trailing edges is normally thought to occur at the wing tips, i.e. at the points of maximum span. However, Newman finds that transition occurs *forward* of the wing tip, at positive angle of attack.

We show here that such a phenomenon is not confined to the low-aspect-ratio limit, and that a result of Newman's, namely that transition occurs where the *mean* (above and below the wing) velocity vector is tangent to the edge, applies generally. However, we also show that, in the linearized case, transition necessarily occurs at the wing tips.

In order to illustrate the nonlinear problem, we give here a semi-numerical solution for a wing of circular planform. In the special case of an exponentially varying clearance, it is possible to write down a solution in the form of a Bessel-function expansion, and good accuracy is achievable by truncating this series to a small number of terms. This solution can act as a test case for computer programs that are devised to solve general classes of extreme ground-effect problems.

2. Derivation of gap-flow boundary-value problems

If we assume steady irrotational motion of an inviscid incompressible fluid, we must determine a velocity potential $\phi = \phi(x, y, z)$ satisfying Laplace's equation

$$\phi_{xx}+\phi_{yy}+\phi_{zz} = 0, \tag{2.1}$$

everywhere in the flow region, subject to suitable boundary conditions. These are that

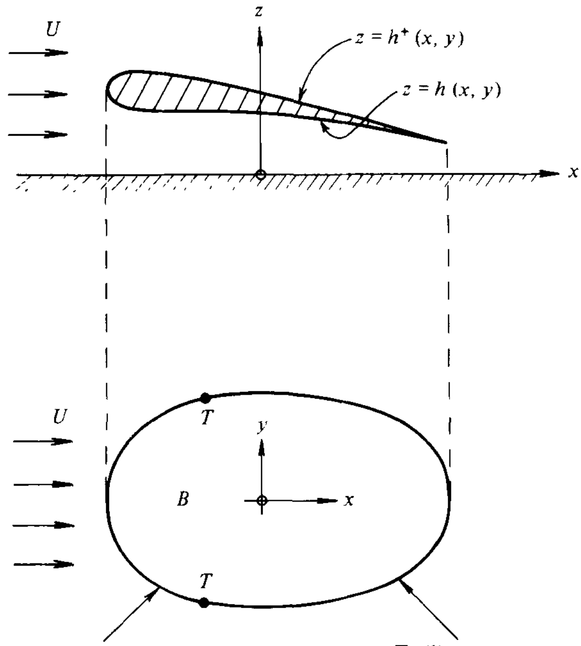

FIGURE 1. Sketch of flow situation and coordinate system.

a uniform stream is recovered at infinity, i.e.

$$\phi \rightarrow Ux \quad \text{at } \infty, \tag{2.2}$$

the plane $z = 0$ is an impermeable ground, i.e.

$$\phi_z = 0 \quad \text{on} \quad z = 0, \tag{2.3}$$

and the body is fixed. If $z = h(x,y)$ is the lower and $z = h^+(x,y)$ the upper surface of the body, this means that

$$\phi_z = \phi_x h_x + \phi_y h_y \quad \text{on} \quad z = h, \tag{2.4}$$

with a similar boundary condition on $z = h^+$. Figure 1 is a sketch of the boundary geometry and coordinate system for this problem.

The above is a well-posed boundary-value problem, but it is *not* the problem of interest for lifting surfaces. We are really interested in bodies that shed vorticity into a trailing wake, and hence flows that do not satisfy (2.1) everywhere. In general, for an arbitrary bluff body, the problem of specifying properties of such a rotational wake is difficult, and in any case not of interest in the present study.

We now assume that the body is *thin*, in the sense that both h and h^+ are small. Then any such wake will also be thin, and we may assume (2.1) to hold *almost everywhere*, i.e. everywhere except within a wake of vanishing thickness, which becomes a vortex sheet. This vortex sheet must have the property that there is no jump in pressure p across it, where p is given by Bernoulli's equation i.e.

$$\frac{p}{\rho} + \tfrac{1}{2}(\phi_x^2 + \phi_y^2 + \phi_z^2) = \frac{p_\infty}{\rho} + \tfrac{1}{2}U^2, \tag{2.5}$$

if p_∞ is the ambient pressure. The wake condition of zero pressure jump then must hold also at points of attachment between wake and body, this being the well-known Kutta condition.

Thus if B denotes the planform of the thin body (i.e. its projection on the plane $z = 0$), so that (2.4) holds for $(x, y) \in B$, and if Γ denotes the perimeter of B, then we can divide Γ into a leading part Γ_L and a trailing part Γ_T. The wake is shed from Γ_T, and the Kutta condition asserts that the value of p given by (2.5) is continuous between upper and lower surfaces at all points of Γ_T.

The final question in problem specification concerns the distinction between Γ_L and Γ_T, i.e. the choice of two or more transition points between leading and trailing edges. Such transition points may be fixed *a priori* by the geometry of the planform B, especially if its tips or extremities in the y-direction are sharp points. Otherwise, the flow itself determines the leading–trailing transition, in such a way that (in broad terms) the flow mean is *inward* to B for all points of Γ_L and *outward* from B for all points of Γ_T. The actual procedure for determining transition is discussed later.

Now if $h, h^+ = O(\epsilon L)$ and $\epsilon \rightarrow 0$, where L is an x-wise lengthscale, the net disturbance to the uniform stream vanishes almost everywhere, and we may write

$$\phi = Ux + O(\epsilon). \tag{2.6}$$

The exception to (2.6) is for points within the small gap between body and ground, i.e. for $(x, y) \in B$, $0 < z < h(x, y)$. In this gap region, $\nabla\phi$ is not necessarily close to U, since very small gaps can induce large velocities.

However, the smallness of the gaps does tend to prevent z-wise fluid motion, and we can expand in a Taylor series

$$\phi(x, y, z) = \phi(x, y, 0) - \tfrac{1}{2}z^2[\phi_{xx}(x, y, 0) + \phi_{yy}(x, y, 0)] + O(\epsilon^4), \tag{2.7}$$

which satisfies (2.1) and (2.3), and also satisfies (2.4) if

$$\frac{\partial}{\partial x}\left(h\frac{\partial\phi}{\partial x}\right) + \frac{\partial}{\partial y}\left(h\frac{\partial\phi}{\partial y}\right) = 0, \tag{2.8}$$

where $\phi = \phi(x, y)$ is used instead of $\phi(x, y, 0)$. Equation (2.8) must be satisfied for $(x, y) \in B$, subject to

$$\phi = Ux \quad \text{on } \Gamma_L, \tag{2.9}$$

$$\phi_x^2 + \phi_y^2 = U^2 \quad \text{on } \Gamma_T. \tag{2.10}$$

Equation (2.9) matches the gap-region flow to the exterior flow (2.6) at the leading edge. The detailed matching (see Tuck & Bentwich 1983) involves a local entrance flow with a stagnation point on the lower surface of the wing, at a small $O(h)$ distance from the leading edge. Equation (2.10) is a consequence of the Kutta condition, using (2.5).

Thus the gap-flow problem reduces to a two-dimensional nonlinear mixed boundary-value problem on B for the elliptic partial differential equation (2.8), subject to boundary conditions (2.9), (2,10) on the boundary of B. Note that, if $h =$ constant, this problem has the solution

$$\phi \equiv Ux \quad \text{in } B. \tag{2.11}$$

That is, a *uniform* gap does not disturb the incident stream. There is only a non-trivial disturbance within the gap if h is non-constant. This is a conclusion subject to an $O(\epsilon)$ error; that is, we are neglecting all velocity perturbations that tend to zero with the gap size.

If the boundary Γ possesses slope discontinuities near its y-wise extremities, these will tend to pre-determine the transition point between Γ_L and Γ_T. Let us for the moment assume that this is *not* the case, i.e. that Γ is a smooth curve in a region where transition is possible. Then, if β denotes the angle between Γ and the x-axis, (2.9) is equivalent to

$$\phi_t = U\cos\beta \quad \text{on } \Gamma_{\mathrm{L}}, \tag{2.12}$$

where $\partial/\partial t$ denotes differentiation tangent to Γ. Similarly (2.10) can be written

$$\phi_t^2+\phi_n^2 = U^2 \quad \text{on } \Gamma_{\mathrm{T}}, \tag{2.13}$$

where $\partial/\partial n$ is the derivative normal to Γ in the (x, y)-plane. If we demand that the tangential velocity ϕ_t be continuous across the transition point, this means that, at that point,

$$\phi_n = \pm U\sin\beta. \tag{2.14}$$

Of the two possibilities (2.14), the + sign is correct. If we were to choose the − sign, this would mean that $\nabla\phi = U\boldsymbol{i}$ at transition; i.e. the under-vehicle velocity is exactly the same as the uniform stream, and hence the average of under- and over-vehicle velocities in general possesses a non-zero inward or outward component. However, if we choose

$$\phi_n = U\sin\beta, \tag{2.15}$$

at transition, then the *mean* component of velocity normal to the planform is *zero*, the normal component of the above-vehicle velocity being $-U\sin\beta$. The choice (2.15) means that the local flow streamline at transition makes an angle 2β with the x-axis, so that, when averaged with the x-directed flow of the same magnitude *above* the vehicle, we obtain local mean tangency to Γ at transition.

3. Skirts or infinite aspect ratio

The boundary-value problem (2.8)–(2.10) can be solved completely if there is no dependence on y. That is, if $h = h(x)$ and $\phi = \phi(x)$ only, and Γ_{L} is the point $x = 0$ while Γ_{T} is at $x = L$, then

$$\phi_x = \frac{Uh(L)}{h(x)}, \tag{3.1}$$

satisfies (2.8) and (2.10). So also does the negative of (3.1), but we reject that solution, since it is inward rather than outward to the trailing edge. Equation (3.1) can be integrated with respect to x, subject to the initial condition (2.9), to determine finally the velocity potential ϕ. However, for lift computation purposes, the velocity (3.1) is adequate, and (2.9) is not needed.

This two-dimensional problem has been the subject of a number of studies, e.g. Tuck (1978, 1980, 1981, 1982; Tuck & Bentwich 1983). The resulting lift has an upper bound or maximum upward force of $\frac{1}{2}\rho U^2 L$ per unit span, corresponding to uniform stagnation pressure $p_\infty+\frac{1}{2}\rho U^2$ beneath the gap, and attained when the trailing edge just touches the ground.

On the other hand, there is no lower bound. That is, in principle, the *downward* force or negative lift can be made as large as one pleases, by allowing $h(x)$ to become vanishingly small at some $x = x_0 < L$, while keeping $h(L) > 0$. This Venturi effect is the reason for the use of ground effect on racing cars. It is limited only by real-fluid considerations, associated with the need to prevent too-early separation in the effective 'diffuser' generated between the minimum-gap station $x = x_0$ and the trailing edge $x = L$.

That is, of course, providing the basic two-dimensional assumption is valid. This means either a vehicle of impossibly high aspect ratio, i.e. very wide compared to its length, or else some artificial means of ensuring two-dimensionality. It is clear that, unless one of these requirements is met, not only will two-dimensional flow not be achieved, but the benefits of negative-lift ground effect will be lost. That is, the more we seek to achieve (relative to p_∞) a negative pressure, by letting $h \to 0$ at some $x = x_0$, the greater will be the tendency for this negative pressure to suck air in *sideways*, so destroying the two-dimensional flow assumption.

The solution achieved during the late 70s in Formula I racing was to use so-called *skirts* at the extreme sides of the vehicle. These are solid curtains, completely blocking the gap by extending right down to the ground; indeed, in most applications, actually dragging along the ground.

The solution (3.1) applies whenever $h = h(x)$ alone. That is, providing the car is designed so that every (lateral) section is identical, a two-dimensional flow is a possible solution. If the width is infinite, we need not worry about the sides, of course. But, if the width is finite, the solution must be such that there is no flow through these sides, i.e. with skirts parallel to the x-axis, $\phi_y = 0$ at the sides. Since the solution (3.1) has this property, it applies exactly to straight-skirted vehicles, where the gap is the same at all lateral stations.

In practice, the beneficial effect of skirts is available whether or not exact two-dimensionality, as defined above, is achieved. The general and obvious idea is to reduce the gap to a very small value at some intermediate point, and then allow it to increase (gently) toward the trailing edge. This will cause a beneficial low pressure, and skirts help to retain that pressure against self-destruction from the side.

The general problem (2.8)–(2.10) can be modified to include skirts, simply by subdividing the contour Γ into portions Γ_L, Γ_T and Γ_S, instead of just Γ_L and Γ_T as before. Now Γ_S is a 'no-flow' region, subject to the boundary condition

$$\phi_n = 0 \quad \text{on } \Gamma_S. \tag{3.2}$$

That is, the boundary-value problem (2.8)–(2.10), (3.2) describes flow beneath a thin wing of arbitrary aspect ratio containing arbitrarily prescribed skirts along Γ_S. In general, we cannot expect velocity continuity at the ends of Γ_S, and indeed in theory an inviscid fluid moves around the sharp ends of a finite-length skirt at infinite velocity. However, there must be a particular choice of location for the skirt, such that the flow stays bounded in the limit as the length of the skirt tends to zero, and this limiting point is just the smooth transition point discussed in §2.

Another respect in which the theory of two-dimensional ground effect differs markedly from three-dimensional theory, concerns extensions and projections of the flow, and the ensuing wake vortex sheet. If we start with a given two-dimensional body for $0 < x < L$, and add to that body projections *of constant clearance*, either forward or aft of the original body, the pressure on such projections is constant and equal to that at $x = 0$ or $x = L$ respectively.

For example, in studying negative-lift ground effect on skirted vehicles, one need concern oneself (in the main) only with the *expanding* portion of the gap, assuming that the large negative pressure at the minimum-clearance point will be felt not only at that point, but also at all stations further forward, providing the clearance is sensibly constant at such stations. This is no longer the case at finite aspect ratio, or when skirts are removed.

Similarly, a rearward projection at constant clearance has no effect on the net lift, since the pressure in such an extended channel must be equal to the freestream pressure. Indeed, such an extension is fluid-dynamically indistinguishable from the wake. In the two-dimensional case (Tuck 1980) the wake is necessarily of constant height equal to that of the trailing edge, and the flow velocity in the wake is equal to the freestream value, so that the vortex-sheet strength is zero – in effect, there is no vortex sheet present. However, in the three-dimensional case (and also in unsteady flow, see Tuck 1978) there is a vortex sheet of non-uniform strength located at a non-constant height and in a region of non-constant lateral extent. All these quantities must be determined by, in effect, solving (2.8) and (2.10) together as coupled equations, for unknowns $\phi(x, y)$ and $h(x, y)$. No attempt is made here to carry out such computations.

4. The linearized problem

Since (2.11) applies if $h = h_0 =$ constant, we may expect that, if h is nearly constant, the gap flow is nearly of the form (2.11), i.e. nearly undisturbed by the body. That is, if we write

$$h(x,y) = h_0 + h_1(x,y), \tag{4.1}$$

where $h_1 \ll h_0$, then

$$\phi(x,y) = Ux + \phi_1(x,y), \tag{4.2}$$

where $\phi_1 \ll Ux$. Substituting into (2.8)–(2.10) and retaining only first-order terms in h_1 and ϕ_1, we obtain

$$\frac{\partial^2 \phi_1}{\partial x^2} + \frac{\partial^2 \phi_1}{\partial y^2} = -\frac{U}{h_0}\frac{\partial h_1}{\partial x}, \tag{4.3}$$

to be solved subject to

$$\phi_1 = 0 \quad \text{on } \Gamma_{\mathrm{L}}, \tag{4.4}$$

$$\phi_{1x} = 0 \quad \text{on } \Gamma_{\mathrm{T}}. \tag{4.5}$$

The problem (4.3)–(4.5) is a linearized equivalent of (2.8)–(2.10). Equation (4.3) is a Poisson equation for ϕ_1, with a known right-hand side involving the specified gap perturbation $h_1(x, y)$. The boundary condition (4.5) is, in contrast with (2.10), linear, but still somewhat awkward, in that it involves an 'oblique' derivative $\partial/\partial x$ which is neither normal nor tangent to the boundary Γ_{T} in general.

It is instructive to examine the effects of linearization at the transition point between Γ_{L} and Γ_{T}. Since

$$\frac{\partial}{\partial x} = \cos\beta\frac{\partial}{\partial t} - \sin\beta\frac{\partial}{\partial n}, \tag{4.6}$$

assuming continuity of ϕ_{1t}, (4.4), (4.5) are compatible only if

$$\sin\beta\,\phi_{1n} = 0, \tag{4.7}$$

This can be satisfied either if $\sin\beta = 0$ or if $\phi_{1n} = 0$, and it is not easy to decide, on the basis of the *a priori*-linearized problem, which of these alternatives applies.

However, if we return temporarily to the nonlinear problem near the transition point, it is clear that the only alternative compatible with (2.15) is that $\sin\beta = 0$. That is, if we substitute (4.2) into (2.15), we find that, formally,

$$\phi_{1n} = 2U\sin\beta, \tag{4.8}$$

at transition. This is consistent with the linearization $\phi_1 \ll Ux$ only if (to leading order) $\sin\beta = 0$. Physically if the local flow is required to make an angle 2β with the x-axis at transition, linearization is valid only if that angle 2β is vanishingly small. Note that the alternative conclusion $\phi_{1n} = 0$ would apply if we had chosen the sign in (2.14).

That is, we have shown that, in the linearized problem, smooth leading-to-trailing-edge transition always occurs at a point where the planform boundary is locally parallel to the incident stream. Normally (e.g. for convex planforms) there will only be two such points and these will mark the *widest* points of the body, its lateral extremities or wing tips. Indeed, this is what we should anticipate from conventional aerodynamics. However, it is no longer true in the nonlinear case.

The linear problem (4.3)–(4.5) was first discussed by Widnall & Barrows (1970), who obtained an explicit solution for a 'semi-ellipitical' planform with a uniformly sloping bottom, i.e. with the right-hand side of (4.3) constant. That is, Widnall & Barrows' solution has a straight trailing edge $x = 0$, and a leading edge in $x < 0$, whose contour is a semi-ellipse.

In principle, it is possible to obtain numerical solutions of (4.3)–(4.5) for arbitrary input $h(x, y)$ and planform Γ. However, in view of the mixed nature of the boundary conditions (4.4), (4.5), such numerical methods will inevitably involve matrix inversion or iteration. Hence there seems little point in pursuing the linearized version of this problem further, when the full equations (2.8)–(2.10) are only slightly more difficult to handle numerically.

5. Circular planforms

In this section we present a seminumerical solution of the full problem (2.8)–(2.10), for the special case when

$$h(x, y) = h_0 \mathrm{e}^{-2kx}, \tag{5.1}$$

for some constants h_0, k, and when Γ is the circle

$$x^2 + y^2 = a^2. \tag{5.2}$$

Although circular planforms and exponential clearances are here chosen for analytical convenience, the results are not untypical of what would be obtained for more-practical geometries.

If h is given by (5.1), then (2.8) becomes

$$\phi_{xx} - 2k\phi_x + \phi_{yy} = 0, \tag{5.3}$$

so that

$$\Phi = \mathrm{e}^{-kx}\phi \tag{5.4}$$

satisfies a Helmholtz-type equation

$$\nabla^2\Phi = k^2\Phi. \tag{5.5}$$

If we use polar coordinates (r, θ) to solve (5.5) by separation, we can represent the solution to (5.3) in the form

$$\phi(r, \theta) = \mathrm{e}^{kr\cos\theta} \sum_{j=0}^{\infty} \gamma_j \frac{I_j(|k|\, r)}{I_j(|k|\, a)} \cos j\theta, \tag{5.6}$$

for some coefficients γ_j to be determined, where I_j is the modified Bessel function of the first kind.

If $u = \phi_r$ and $v = \phi_\theta / r$ are the velocity components in these coordinates, we can write

$$\phi = \sum_{j=0}^{\infty} \gamma_j \phi_j(r, \theta), \tag{5.7}$$

$$u = \sum_{j=0}^{\infty} \gamma_j u_j(r, \theta), \tag{5.8}$$

$$v = \sum_{j=0}^{\infty} \gamma_j v_j(r, \theta), \tag{5.9}$$

where

$$\phi_j = \mathrm{e}^{kr\cos\theta}\frac{I_j(|k|\,r)}{I_j(|k|\,a)}\cos j\theta, \tag{5.10}$$

$$u_j = k\cos\theta\,\phi_j + |k|\,\mathrm{e}^{kr\cos\theta}\frac{I_j'(|k|\,r)}{I_j(|k|\,a)}\cos j\theta, \tag{5.11}$$

$$v_j = -k\sin\theta\,\phi_j - \frac{j}{r}\mathrm{e}^{kr\cos\theta}\frac{I_j(|k|\,r)}{I_j(|k|\,a)}\sin j\theta. \tag{5.12}$$

The boundary conditions (2.9), (2.10) can now be written

$$E_\theta(\gamma_j) = 0, \tag{5.13}$$

where

$$E_\theta = \begin{cases} \phi - Ux & (\theta_{\mathrm{T}} \leqslant \theta \leqslant \pi), \\ u^2+v^2-U^2 & (0 \leqslant \theta \leqslant \theta_{\mathrm{T}}), \end{cases} \tag{5.14}$$

evaluated on $r = a$, and $\theta = \theta_{\mathrm{T}}$ is the transition position, to be determined. For the moment, suppose we assume θ_{T} is known, and force (5.13) to hold at a fixed set of N θ-values, $\theta = \theta_i$, $i = 0, 1, 2, \ldots, N-1$. Then, if we truncate the series (5.6) etc. to N coefficients, (5.13) is a set of N nonlinear equations in N unknowns γ_j, $j = 0, 1, 2, \ldots, N-1$.

This set can be solved by Newton iteration, i.e. replace γ_j by $\gamma_j + \delta\gamma_j$, where

$$\sum_{j=0}^{N-1}\left(\frac{\partial E_{\theta i}}{\partial \gamma_j}\right)\delta\gamma_j = -E_{\theta i}. \tag{5.15}$$

The elements of the matrix in (5.15) can be evaluated explicitly from (5.14); thus

$$\frac{\partial E_{\theta i}}{\partial \gamma_j} = \begin{cases} \phi_j & (\theta_{\mathrm{T}} \leqslant \theta_i \leqslant \pi), \\ 2uu_j + 2vv_j & (0 \leqslant \theta_i \leqslant \theta_{\mathrm{T}}), \end{cases} \tag{5.16}$$

evaluated with $r = a$, $\theta = \theta_i$. The coefficients ϕ_j, u_j, v_j given by (5.10)–(5.12) take especially simple forms at $r = a$, and, in particular, only u_j involves Bessel functions, via the ratio I_j'/I_j. For fixed θ_{T}, we need merely choose a trial set of γ_j, solve the linear equations (5.15), and so iteratively improve γ_j.

In practice, we must also iteratively improve our knowledge of the transition point $\theta = \theta_{\mathrm{T}}$. This is most easily done by keeping track of the mean normal velocity, as specified by (2.15) (with $\beta = \theta - \frac{1}{2}\pi$ in the present case), i.e.

$$\bar{u}_n = u + Ua\cos\theta, \tag{5.17}$$

while evaluating the matrix elements (5.16). So long as $\bar{u}_n < 0$, we are in the leading-edge regime $\theta_{\mathrm{T}} < \theta_i \leqslant \pi$, whereas when $\bar{u}_n > 0$ we are in the trailing-edge regime $0 \leqslant \theta_i < \theta_{\mathrm{T}}$. At each iteration step, we record the value of i such that $\bar{u}_n$ changes sign between θ_i and θ_{i+1}, and determine θ_{T} by linear interpolation in this range.

A suitable first guess for the coefficients is that corresponding to the linearization in §4. That is, if we let $k \to 0$, the solution (5.6) is dominated by the uniform stream $\phi = Ur\cos\theta$ only if $\gamma_1 \to 1$ and $\gamma_j \to 0$ for all $j \neq 1$. Thus the choice $\gamma_1 = 1$, $\gamma_j = 0$, $j \neq 1$ is very accurate for small k, and adequate as a starting point for all k. In practice, no more than 5 iterations are needed to reduce E_{θ_i} below 10^{-4}. Values of N up to 20 were tried, but in fact $N = 10$ is adequate for 2–3-figure accuracy. For example, the early coefficients $\gamma_1, \gamma_2, \gamma_3$ do not change by more than 0.001 beyond $N = 10$, and

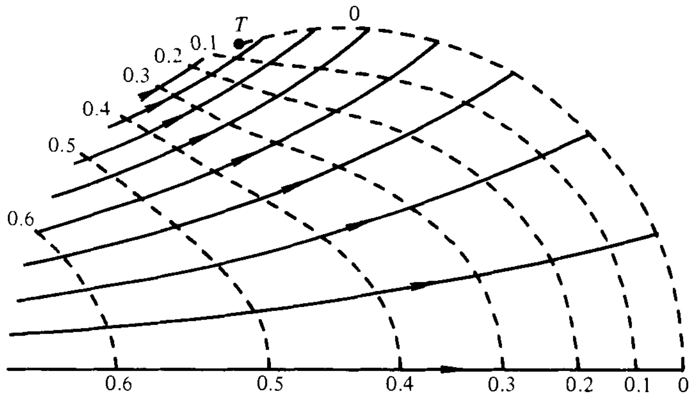

FIGURE 2. Streamlines (solid) and constant-pressure contours (dashed) for a wing of circular planform. Ground clearance is proportional to $\exp(-2kx)$, with $ka = 0.5$, i.e. positive angle of attack. Leading-edge to trailing-edge transition occurs at the point T.

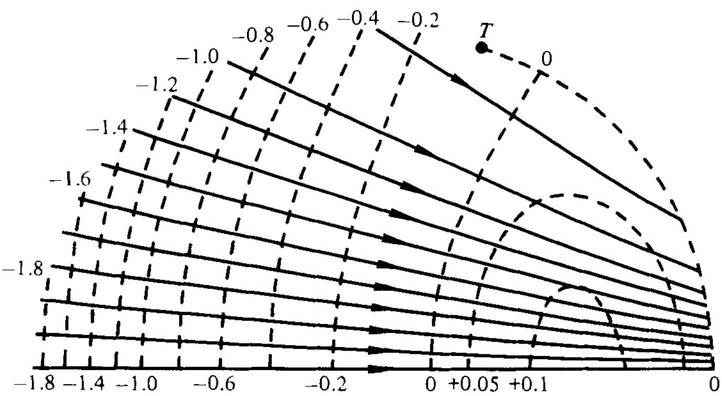

FIGURE 3. Same as figure 2, with $ka = -0.5$, i.e. negative angle of attack.

the coefficients γ_{10}, γ_{11} etc. are smaller than 0.0002. Similarly, final values of θ_T, of the maximum pressure, and of the net lift force vary by less than 0.1 % beyond $N = 10$.

Figures 2 and 3 show streamlines (solid) and constant pressure contours (dashed) for two highly nonlinear cases, $ka = +0.5$ and $ka = -0.5$ respectively. The contraction ratio $h(a)/h(-a)$ takes the value $\exp(4ka) = 7.4$ in figure 2, and its reciprocal in figure 3.

Thus figure 2 corresponds to positive angle of attack, with positive pressures and a net upward lift force. Transition is at $\theta_T = 107°$, forward of the lateral extremity. The positive pressure tends to drive some fluid outward sideways. Recall that the flow shown beneath the body is accompanied by a wake, whose properties we are not attempting to compute, but which will in some sense represent a continuation of the streamlines shown. In the nonlinear case, there is no reason to believe that the wake will have edges parallel to the uniform stream, and there is a strong indication from figure 2 that, in the case of positive angle of attack, it will spread out behind the body.

In figure 3 the effective angle of attack is negative, and quite large negative pressures are induced at the narrowest point near the extreme leading edge. However, there are (small) positive pressures in the rearward quarter. Transition is at 72°, aft of the lateral extremity. Streamlines are generally inward from the sides, indicating

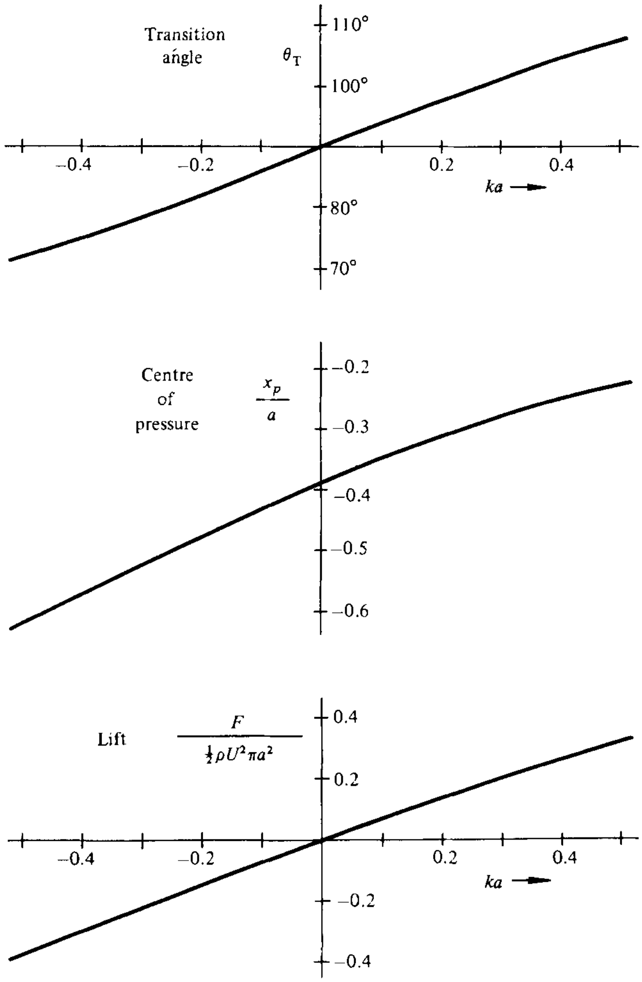

FIGURE 4. Plots as functions of ka, of the leading-edge to trailing-edge transition angle, the centre-of-pressure location, and the net lift coefficient.

suction of fluid by the negative pressures. Because of this, the actual values of the negative pressure are not nearly as great as would occur in a two-dimensional flow with a similar clearance distribution. For example, a rectangular skirted body with the same $h(x)$ would produce a minimum pressure coefficient of $1-e^4 = -54$ instead of the value -2 achieved here.

Figure 4 shows various output quantities as a function of k. The net lift force and moment are computed by numerical pressure integration. Results for small $|k|$ vary linearly with k, and have been checked using the methods of §4. This is equivalent to a small-ka approximation to the equations used in the present section, with Bessel functions replaced by appropriate powers of kr, the boundary conditions (5.13) linearized, and $\theta_T = \frac{1}{2}\pi$. All coefficients γ_j tend to zero to leading order, except $\gamma_1 \rightarrow Ua$. The linearized lift is then proportional to the first correction $\gamma_1 - Ua$ to this coefficient, and we find that the lift coefficient (force/$\frac{1}{2}\rho U^2 \pi a^2$) varies as $0.727ka$, and is centred at $x/a = -0.390$. The linear variation of lift with ka persists substantially out to $ka = \pm 0.5$, in spite of the nonlinear character of the flows depicted in figures 2 and 3.

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